

## STRESS AND DEFORMATIONS IN TOROIDAL SHELLS WITH ELLIPTICAL TRANSVERSAL SECTION

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**Abstract.** The article approaches the analysis of stress and deformation states; in the toroidal shell with the elliptical cross section with two specific positions, relative to the axis of symmetry. The internal pressure of a working environment as well as the effect of its temperature are considered external loads. The two effects may overlap taking into account the elastic state of the shell material.

**Keywords:** Stress, deformations, toroidal shells, external loads

### 1. INTRODUCTION

The industrial equipment's in the structure of the process technology, among other forms of shells, also contains toroidal elliptical cross-sections with large semiax along the axis of symmetry or perpendicular to it. These structures are used, as well as those with a circular cross-section, with the advantages recognized in terms of economics of building materials or diminution of the stress concentration by connecting with other geometrical shapes of envelopes [1–14]. In addition to the stable application in the elastic field of application, the literature also considers the elastic or plastic instability of such structures [9-11].

### 2. TOROIDAL SHELL WITH ELLIPTICAL TRANSVERSAL SECTION

#### 2.1. The main curvature rays

When replacing with an ellipse the circle which can generate a toroidal surface with large semiax along the  $Ox$  axis and originated in the center of the ellipse and the small semiax along the  $Oy$  axis and also originated in the center of the ellipse, the main radii of curvature are form as [15]:

$$R_1 = \left[ 1 + \left( \frac{dr}{dy} \right)^2 \right]^{3/2} / \left( d^2r / dy^2 \right); R_2 = r / |\sin \varphi| \quad (1)$$

**Note:** In order to preserve the physical significances of the calculated sizes (main radii of curvature, stresses and deformations), the following will be accepted  $|\sin \varphi|$  although for the contour  $\overline{ABC}$  of the ellipse the value of the function is positive, while for the contour  $\overline{CDA}$  it has a negative value (the portion  $\overline{DA}$ ).

The current radius of an ellipse point is determined by the formula [15]:

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$$r = R_0 \pm \sqrt{a^2 \cdot (1 - y^2/b^2)} \tag{2}$$

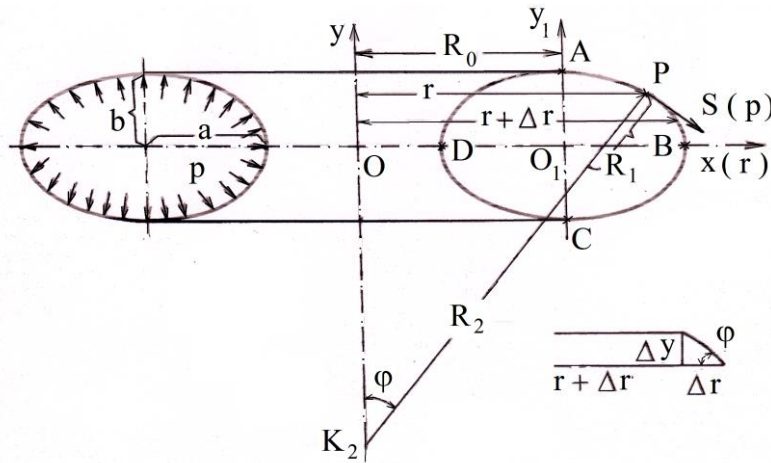


Fig. 1. Toroidal shell with elliptical cross-section with large horizontal axis ( $a > b$ ) [15, 16].

The (+) sign characterizing the semi ellipse  $\widehat{ABC}$ , and the (-) sign semi ellipse  $\widehat{CDA}$ . In this way, the current  $x$  quota will be accepted with a positive value in the evaluation of the distance between the current radius  $r$  and the position of the center of the ellipse defined by  $R_0$ . After crossing the arc  $\widehat{AB}$ , the point  $P$  on route  $\widehat{BC}$  is defined by the existence of the center  $K_2$ , which is above the straight  $OO_1$  (mirror image). As a result, symmetric  $P$  points relative to  $OO_1$  will be at the same distance  $x$ , measured with respect to the vertical axis  $O_1y_1$ . A similar position can be specified for the current  $P$  points, which pass the route  $\widehat{CDA}$ . The following observations can be made [15]:

- for contour  $\widehat{ABC}$ :

$$dr/dy = - \left[ a \cdot \sqrt{a^2 - (r - R_0)^2} \right] / \left[ b \cdot (r - R_0) \right] \tag{3}$$

$$d^2r/dy^2 = - a^4 / \left[ b^2 \cdot (r - R_0)^3 \right] \tag{4}$$

- for contour  $\widehat{CDA}$ :

$$dr/dy = \left[ a \cdot \sqrt{a^2 - (R_0 - r)^2} \right] / \left[ b \cdot (R_0 - r) \right] \tag{5}$$

$$d^2r/dy^2 = a^4 / \left[ b^2 \cdot (R_0 - r)^3 \right] \tag{6}$$

Appropriate customizations lead to:

- for contour  $\widehat{ABC}$ :

$$\{ \varphi \in [0; \pi]; r \geq R_0; r = R_0 + x; x \in [0; a]; A(\varphi=0); B(\varphi=\pi/2); C(\varphi=\pi) \}:$$

$$R_1 = \left[ (b^2 - a^2) \cdot (r - R_0)^2 + a^4 \right]^{3/2} / (a^4 \cdot b) = a^2 \cdot b^2 / (a^2 \cdot \sin^2 \varphi + b^2 \cdot \cos^2 \varphi)^{3/2} =$$

$$= \left[ (b^2 - a^2) \cdot x^2 + a^4 \right]^{3/2} / (a^4 \cdot b) \quad (7)$$

$$R_2 = r \cdot \left[ (b^2 - a^2) \cdot (r - R_0)^2 + a^4 \right]^{1/2} / [b \cdot (r - R_0)] =$$

$$= [R_0 / |\sin \varphi|] + a^2 / (a^2 \cdot \sin^2 \varphi + b^2 \cdot \cos^2 \varphi)^{1/2} =$$

$$= (R_0 + x) \cdot \left[ (b^2 - a^2) \cdot x^2 + a^4 \right]^{1/2} / (b \cdot x) \quad (8)$$

$$\sin \varphi = [b \cdot (r - R_0)] / \left[ (b^2 - a^2) \cdot (r - R_0)^2 + a^4 \right]^{1/2} =$$

$$= [b \cdot (R_2 \cdot |\sin \varphi| - R_0)] / \left[ (b^2 - a^2) \cdot (R_2 \cdot |\sin \varphi| - R_0)^2 + a^4 \right]^{1/2} =$$

$$= b \cdot x / \left[ (b^2 - a^2) \cdot x^2 + a^4 \right]^{1/2} \quad (9)$$

$$\cos \varphi = \left\{ a \cdot [a^2 - (r - R_0)^2]^{1/2} \right\} / \left[ (b^2 - a^2) \cdot (r - R_0)^2 + a^4 \right]^{1/2} =$$

$$= \left\{ a \cdot [a^2 - (R_2 \cdot |\sin \varphi| - R_0)^2]^{1/2} \right\} / \left[ (b^2 - a^2) \cdot (R_2 \cdot |\sin \varphi| - R_0)^2 + a^4 \right]^{1/2} = \quad (10)$$

$$= [a \cdot (a^2 - x^2)^{1/2}] / \left[ (b^2 - a^2) \cdot x^2 + a^4 \right]^{1/2}$$

- for contour CDA:

$$\{ \varphi \in [\pi; 2 \cdot \pi]; r \leq R_0; x \in [0; a]; C(\varphi = \pi); D(\varphi = 3 \cdot \pi / 2); A(\varphi = 2 \cdot \pi) \}:$$

$$R_1 = \left[ a^4 + (b^2 - a^2) \cdot (R_0 - r)^2 \right]^{3/2} / (a^4 \cdot b) = a^2 \cdot b^2 / (a^2 \cdot \sin^2 \varphi + b^2 \cdot \cos^2 \varphi)^{3/2} =$$

$$= \left[ a^4 + (b^2 - a^2) \cdot x^2 \right]^{3/2} / (a^4 \cdot b) \quad (11)$$

$$R_2 = r \cdot \left[ a^4 + (b^2 - a^2) \cdot (R_0 - r)^2 \right]^{1/2} / [b \cdot (R_0 - r)] =$$

$$= [R_0 / |\sin \varphi|] - a^2 / (a^2 \cdot \sin^2 \varphi + b^2 \cdot \cos^2 \varphi)^{1/2} = \quad (12)$$

$$= (R_0 - x) \cdot \left[ a^4 + (b^2 - a^2) \cdot x^2 \right]^{1/2} / (b \cdot x)$$

$$\sin \varphi = [b \cdot (R_0 - r)] / \left[ a^4 + (b^2 - a^2) \cdot (R_0 - r)^2 \right]^{1/2} =$$

$$= [b \cdot (R_0 - R_2 \cdot |\sin \varphi|)] / \left[ a^4 + (b^2 - a^2) \cdot (R_0 - R_2 \cdot |\sin \varphi|)^2 \right]^{1/2} =$$

$$= b \cdot x / \left[ a^4 + (b^2 - a^2) \cdot x^2 \right]^{1/2} \quad (13)$$

$$\cos \varphi = \left\{ a \cdot [a^2 - (R_0 - r)^2]^{1/2} \right\} / \left[ a^4 + (b^2 - a^2) \cdot (R_0 - r)^2 \right]^{1/2} =$$

$$= \left\{ a \cdot \left[ a^2 - (R_0 - R_2 \cdot |\sin \varphi|)^2 \right]^{1/2} \right\} / \left[ a^4 + (b^2 - a^2) \cdot (R_0 - R_2 \cdot |\sin \varphi|)^2 \right]^{1/2} = \\ = \left[ a \cdot (a^2 - x^2)^{1/2} \right] / \left[ a^4 + (b^2 - a^2) \cdot x^2 \right]^{1/2} \quad (14)$$

where  $R_0$  has the meaning in Figure 1, with equality being noted  $x = r - R_0 \in [0; a]$ .

**Note:** For the determination of dimensions  $x$ , the semi ax  $b$  conveniently divided in a positive or negative sense, the respective values being introduced into equality:

$$x = a \cdot \sqrt{1 - y^2/b^2} \quad (15)$$

deduced from the equation of the generating ellipse (with center  $O_1$  - Figure1).

## 2.2. Normal unitary forces, meridional and annular

Under the action of the internal overpressure  $p$ , the following unit expressions are found for the normal unitary forces  $S(p)$  and  $T(p)$  [15]:

- for contour  $\widehat{ABC}$  ( $r \geq R_0$ ;  $r = R_0 + x$ ;  $x \in [0; a]$ ):

$$S(p) = \frac{p \cdot (r^2 - R_0^2)}{2 \cdot r \cdot (\sin \varphi)} = \frac{p \cdot (r + R_0)}{2 \cdot b \cdot r} \cdot \left[ a^4 + (b^2 - a^2) \cdot (r - R_0)^2 \right]^{1/2} \quad (16)$$

$$T(p) = p \cdot R_2 \cdot \left[ 1 - \frac{r^2 - R_0^2}{2 \cdot R_1 \cdot r \cdot (\sin \varphi)} \right] = \frac{p}{2 \cdot b} \cdot \frac{a^4 + 2 \cdot r \cdot (b^2 - a^2) \cdot (r - R_0)}{\left[ a^4 + (b^2 - a^2) \cdot (r - R_0)^2 \right]^{1/2}} \quad (17)$$

in which the variable  $r$ , respectively:

- for contour  $\widehat{CDA}$  ( $r \leq R_0$ ;  $r = R_0 - x$ ):

$$S(p) = \frac{p \cdot (R_0^2 - r^2)}{2 \cdot r \cdot (\sin \varphi)} = \frac{p \cdot (r + R_0)}{2 \cdot b \cdot r} \cdot \left[ a^4 + (b^2 - a^2) \cdot (R_0 - r)^2 \right]^{1/2} \quad (18)$$

$$T(p) = p \cdot R_2 \cdot \left[ 1 - \frac{R_0^2 - r^2}{2 \cdot R_1 \cdot r \cdot (\sin \varphi)} \right] = \frac{p}{2 \cdot b} \cdot \frac{a^4 + 2 \cdot r \cdot (b^2 - a^2) \cdot (R_0 - r)}{\left[ a^4 + (b^2 - a^2) \cdot (R_0 - r)^2 \right]^{1/2}} \quad (19)$$

## 2.3. Stress state

By referring to the relations of calculation of the normal unitary forces, we can deduce the formulas characteristic of the **normal meridional and annular stresses - no thermal effect** - written in the form:

- For contour  $\widehat{ABC}$  ( $r \geq R_0$ ;  $r = R_0 + x$ ;  $x \in [0; a]$ ):

$$\sigma_{1d}(p) = \frac{p}{2 \cdot b \cdot \delta} \cdot \left\{ \left( 1 + \frac{R_0}{R_0 + x} \right) \cdot \left[ a^4 + (b^2 - a^2) \cdot x^2 \right]^{1/2} \right\} \quad (20)$$

$$\sigma_{2d}(p) = \frac{p}{2 \cdot b \cdot \delta} \cdot \frac{a^4 + 2 \cdot (R_0 + x) \cdot (b^2 - a^2) \cdot x}{\left[ a^4 + (b^2 - a^2) \cdot x^2 \right]^{1/2}} \quad (21)$$

representing the normal stress on the right side of the ellipse, the sizes  $R_1$  and  $R_2$  being able to be replaced with  $x$  dependent expressions (7, 8).

- for contour  $\overline{CDA}$  ( $r \leq R_0$ ;  $r = R_0 - |x|$ ;  $|x| \in [0; a]$ ):

$$\sigma_{1s}(p) = \frac{p}{2 \cdot b \cdot \delta} \cdot \left( 1 + \frac{R_0}{R_0 - x} \right) \cdot \left[ a^4 + (b^2 - a^2) \cdot x^2 \right]^{1/2} \quad (22)$$

$$\sigma_{2s}(p) = \frac{p}{2 \cdot b \cdot \delta} \cdot \frac{a^4 + 2 \cdot (R_0 - x) \cdot (b^2 - a^2) \cdot x}{\left[ a^4 + (b^2 - a^2) \cdot x^2 \right]^{1/2}} \quad (23)$$

representing the normal stress on the left side of the ellipse, the sizes  $R_1$  and  $R_2$  being able to be replaced with  $x$  dependent expressions (7, 8).

**Note:** The following customizations are deduced from the comparison of previous expressions for normal stress:

- in points **A** and **C** ( $x = 0$ ;  $r = R_0$  - Figure 1):

$$\sigma_{1d}(p) = \sigma_{1s}(p) = p \cdot a^2 / (b \cdot \delta); \quad \sigma_{2d}(p) = \sigma_{2s}(p) = p \cdot a^2 / (2 \cdot b \cdot \delta) \quad (24)$$

$$\left[ \sigma_{1d}(p); \sigma_{1s}(p) \right] / \left[ \sigma_{2d}(p); \sigma_{2s}(p) \right] = 2 \quad (25)$$

- in point **B** ( $x = a$ ;  $r = R_0 + a$  - Figure 1):

$$\sigma_{1d}(p) = \frac{p \cdot a}{2 \cdot \delta} \cdot \left( 1 + \frac{R_0}{R_0 + a} \right) \quad (26)$$

$$\sigma_{2d}(p) = \frac{p}{2 \cdot b^2 \cdot \delta} \cdot \left[ a^3 + 2 \cdot (R_0 + a) \cdot (b^2 - a^2) \right] \quad (27)$$

- in point **D** ( $|x| = a$ ;  $r = R_0 - a$  - Figure 1):

$$\sigma_{1d}(p) = \frac{p \cdot a}{2 \cdot \delta} \cdot \left( 1 + \frac{R_0}{R_0 - a} \right)$$

$$\sigma_{2d}(p) = \frac{p}{2 \cdot b^2 \cdot \delta} \cdot \left[ a^3 + 2 \cdot (R_0 - a) \cdot (b^2 - a^2) \right] \quad (28)$$

#### 2.4. Deformation state

The specific linear deformations produced in the case of such a structure, in case of the internal overpressure, are in the form [16]:

- for contour  $\widehat{ABC}$  ( $r \geq R_0$ ;  $r = R_0 + x$ ;  $x \in [0; a]$ ):

$$\varepsilon_{1d}(p) = \frac{p}{2 \cdot E \cdot b \cdot \delta} \cdot \left[ \left( 1 + \frac{R_0}{R_0 + x} \right) \cdot \sqrt{a^4 + (b^2 - a^2) \cdot x^2} - \frac{a^4 + 2 \cdot (R_0 + x) \cdot (b^2 - a^2) \cdot x}{\sqrt{a^4 + (b^2 - a^2) \cdot x^2}} - \nu \cdot \frac{a^4 + 2 \cdot (R_0 + x) \cdot (b^2 - a^2) \cdot x}{\sqrt{a^4 + (b^2 - a^2) \cdot x^2}} \right] \quad (29)$$

$$\varepsilon_{2d}(p) = \frac{p}{2 \cdot E \cdot b \cdot \delta} \cdot \left[ \frac{a^4 + 2 \cdot (R_0 + x) \cdot (b^2 - a^2) \cdot x}{\sqrt{a^4 + (b^2 - a^2) \cdot x^2}} - \nu \cdot \left( 1 + \frac{R_0}{R_0 + x} \right) \cdot \sqrt{a^4 + (b^2 - a^2) \cdot x^2} \right] \quad (30)$$

respectively:

- for contour  $\widehat{CDA}$  ( $r \leq R_0$ ;  $r = R_0 - |x|$ ;  $|x| \in [0; a]$ ):

$$\varepsilon_{1s}(p) = \frac{p}{2 \cdot E \cdot b \cdot \delta} \cdot \left[ \left( 1 + \frac{R_0}{R_0 - x} \right) \cdot \sqrt{a^4 + (b^2 - a^2) \cdot x^2} - \frac{a^4 + 2 \cdot (R_0 - x) \cdot (b^2 - a^2) \cdot x}{\sqrt{a^4 + (b^2 - a^2) \cdot x^2}} - \nu \cdot \frac{a^4 + 2 \cdot (R_0 - x) \cdot (b^2 - a^2) \cdot x}{\sqrt{a^4 + (b^2 - a^2) \cdot x^2}} \right] \quad (31)$$

$$\varepsilon_{2s}(p) = \frac{p}{2 \cdot E \cdot b \cdot \delta} \cdot \left[ \frac{a^4 + 2 \cdot (R_0 - x) \cdot (b^2 - a^2) \cdot x}{\sqrt{a^4 + (b^2 - a^2) \cdot x^2}} - \nu \cdot \left( 1 + \frac{R_0}{R_0 - x} \right) \cdot \sqrt{a^4 + (b^2 - a^2) \cdot x^2} \right] \quad (32)$$

condition  $a^4 + (b^2 - a^2) \cdot x^2 > 0$  being accomplished for  $|x| \in [0; a]$ .

Equations (7) - (15) for the main curvature rays and (29) - (32) for specific linear deformations can be used in the evaluation of the displacements of the points of the meridian of the shell  $u(p)$  and  $w(p)$  respectively the radial displacement  $\Delta r(p)$  written in the form [16]:

- for contour  $\widehat{ABC}$  ( $r \geq R_0$ ;  $r = R_0 + x$ ;  $x \in [0; a]$ ):

$$u_d(p) = \frac{b \cdot x}{\sqrt{a^4 + (b^2 - a^2) \cdot x^2}} \cdot I_d \quad (33)$$

$$w_d(p) = \frac{a \cdot \sqrt{a^2 - x^2}}{\sqrt{a^4 + (b^2 - a^2) \cdot x^2}} \cdot I_d - \frac{p \cdot (R_0 + x)}{2 \cdot E \cdot \delta \cdot b^2 \cdot x} \cdot \left\{ a^4 + 2 \cdot (R_0 + x) \cdot (b^2 - a^2) \cdot x - \nu \cdot \frac{2 \cdot R_0 + x}{R_0 + x} \cdot [a^4 + (b^2 - a^2) \cdot x^2] \right\} \quad (34)$$

$$\Delta r_d(p) = \frac{p \cdot (R_0 + x)}{2 \cdot E \cdot b \cdot \delta \cdot \sqrt{a^4 + (b^2 - a^2) \cdot x^2}} \cdot \left\{ a^4 + 2 \cdot (R_0 + x) \cdot (b^2 - a^2) \cdot x - \nu \cdot \frac{2 \cdot R_0 + x}{R_0 + x} \cdot [a^4 + (b^2 - a^2) \cdot x^2] \right\} \quad (35)$$

- for contour  $\overline{CDA}$  ( $r \leq R_0$ ;  $r = R_0 - |x|$ ;  $|x| \in [0; a]$ ):

$$u_s(p) = \frac{b \cdot x}{\sqrt{a^4 + (b^2 - a^2) \cdot x^2}} \cdot I_s \quad (36)$$

$$w_s(p) = \frac{a \cdot \sqrt{a^2 - x^2}}{\sqrt{a^4 + (b^2 - a^2) \cdot x^2}} \cdot I_s - \frac{p \cdot (R_0 - x)}{2 \cdot E \cdot \delta \cdot b^2 \cdot x} \cdot \left\{ a^4 + 2 \cdot (R_0 - x) \cdot (b^2 - a^2) \cdot x - \nu \cdot \frac{2 \cdot R_0 - x}{R_0 - x} \cdot [a^4 + (b^2 - a^2) \cdot x^2] \right\} \quad (37)$$

$$\Delta r_s(p) = \frac{p \cdot (R_0 - x)}{2 \cdot E \cdot b \cdot \delta \cdot \sqrt{a^4 + (b^2 - a^2) \cdot x^2}} \cdot \left\{ a^4 + 2 \cdot (R_0 - x) \cdot (b^2 - a^2) \cdot x - \nu \cdot \frac{2 \cdot R_0 - x}{R_0 - x} \cdot [a^4 + (b^2 - a^2) \cdot x^2] \right\} \quad (38)$$

in which equality is considered:

$$I_d = \frac{p}{2 \cdot E \cdot a \cdot b^2 \cdot \delta} \cdot \left\{ \begin{aligned} & I_1 + R_0 \cdot I_{2d} - \nu \cdot a^4 \cdot I_3 - 2 \cdot \nu \cdot R_0 \cdot (b^2 - a^2) \cdot I_4 - \\ & - 2 \cdot \nu \cdot (b^2 - a^2) \cdot I_5 - a^8 \cdot R_0 \cdot I_6 - \\ & - a^4 \cdot [a^4 + 2 \cdot R_0^2 \cdot (b^2 - a^2)] \cdot I_7 - \\ & - 4 \cdot R_0 \cdot a^4 \cdot (b^2 - a^2) \cdot I_8 - 2 \cdot a^4 \cdot (b^2 - a^2) \cdot I_9 + \\ & + 2 \cdot \nu \cdot R_0 \cdot a^4 \cdot I_{10} + \nu \cdot a^4 \cdot I_{11} \end{aligned} \right\} \quad (39)$$

$$I_s = \frac{p}{2 \cdot E \cdot a \cdot b^2 \cdot \delta} \cdot \left\{ \begin{aligned} & I_1 + R_0 \cdot I_{2s} - \nu \cdot a^4 \cdot I_3 - 2 \cdot \nu \cdot R_0 \cdot (b^2 - a^2) \cdot I_4 - \\ & - 2 \cdot \nu \cdot (b^2 - a^2) \cdot I_5 - a^8 \cdot R_0 \cdot I_6 + \\ & + a^4 \cdot [a^4 + 2 \cdot R_0^2 \cdot (b^2 - a^2)] \cdot I_7 + \\ & + 2 \cdot a^4 \cdot (b^2 - a^2) \cdot I_9 + 2 \cdot \nu \cdot R_0 \cdot a^4 \cdot I_{10} - \nu \cdot a^4 \cdot I_{11} \end{aligned} \right\} \quad (40)$$

respectively the expressions of indefinite integrals:

$$I_1 = \int_x \frac{[a^4 + (b^2 - a^2) \cdot x^2]^{3/2}}{x \cdot (a^2 - x^2)^{1/2}} \cdot dx \quad (41)$$

$$I_{2d} = \int_x \frac{[a^4 + (b^2 - a^2) \cdot x^2]^{3/2}}{x \cdot (R_0 + x) \cdot (a^2 - x^2)^{1/2}} \cdot dx; \quad I_{2s} = \int_x \frac{[a^4 + (b^2 - a^2) \cdot x^2]^{3/2}}{x \cdot (R_0 - x) \cdot (a^2 - x^2)^{1/2}} \cdot dx \quad (42)$$

$$I_3 = \int_x \frac{[a^4 + (b^2 - a^2) \cdot x^2]^{1/2}}{x \cdot (a^2 - x^2)^{1/2}} \cdot dx; \quad I_4 = \int_x \frac{[a^4 + (b^2 - a^2) \cdot x^2]^{1/2}}{(a^2 - x^2)^{1/2}} \cdot dx \quad (43)$$

$$I_5 = \int_x \frac{x \cdot [a^4 + (b^2 - a^2) \cdot x^2]^{1/2}}{(a^2 - x^2)^{1/2}} \cdot dx \quad (44)$$

$$I_6 = \int_x \frac{dx}{x^2 \cdot (a^2 - x^2)^{1/2} \cdot [a^4 + (b^2 - a^2) \cdot x^2]^{1/2}} \quad (45)$$

$$I_7 = \int_x \frac{dx}{x \cdot (a^2 - x^2)^{1/2} \cdot [a^4 + (b^2 - a^2) \cdot x^2]^{1/2}} \quad (46)$$

$$I_8 = \int_x \frac{dx}{(a^2 - x^2)^{1/2} \cdot [a^4 + (b^2 - a^2) \cdot x^2]^{1/2}} \quad (47)$$

$$I_9 = \int_x \frac{x}{(a^2 - x^2)^{1/2} \cdot [a^4 + (b^2 - a^2) \cdot x^2]^{1/2}} \cdot dx \quad (48)$$

$$I_{10} = \int_x \frac{[a^4 + (b^2 - a^2) \cdot x^2]^{1/2}}{x^2 \cdot (a^2 - x^2)^{1/2}} \cdot dx; \quad I_{11} = \int_x \frac{[a^4 + (b^2 - a^2) \cdot x^2]^{1/2}}{x \cdot (a^2 - x^2)^{1/2}} \cdot dx \quad (49)$$

The correspondence that was considered in the calculating the displacement of the points of the meridian of the shell  $u(p)$  and  $w(p)$  and the radial displacement  $\Delta r(p)$ :

$$d\varphi = \frac{a^3 \cdot b}{[a^4 + (b^2 - a^2) \cdot x^2] \cdot (a^2 - x^2)^{1/2}} \cdot dx. \quad (50)$$

**Note:** In the above relationships, the established condition is  $x \neq \pm a$ .

The expressions for **radial linear displacement** of the points of the parallel circles, respectively the **rotation of the meridian**, considering the equalities (16) can also be determined:



$$\Delta r ( p ) = \left\{ u ( p ) \cdot \left[ a \cdot ( a^2 - x^2 )^{1/2} \right] - w ( p ) \cdot b \cdot x \right\} \cdot \frac{1}{\left[ a^4 + ( b^2 - a^2 ) \cdot x^2 \right]^{1/2}} \quad (51)$$

$$\theta ( p ) = \frac{a^4 \cdot b}{\left[ a^4 + ( b^2 - a^2 ) \cdot x^2 \right]^{3/2}} \cdot \left\{ \frac{u ( p ) + \left[ a^4 + ( b^2 - a^2 ) \cdot x^2 \right] \cdot ( a^2 - x^2 )^{1/2}}{b \cdot x^3} \cdot \frac{d w ( p )}{d x} \right\} \quad (52)$$

for the right and the left side, respectively, of the ellipsoidal cross-section of the toroidal shell.

**Note:** Another way of positioning the cross-section of the ellipsoidal shape of the toroidal shell is shown in Figure 2.

The paper [8] identifies appropriate relationships that can assess critical overpressure, written as:

- for the cross-section case, with the large horizontal axis (Figure 1):

$$\left( p_{cr} \right)_1 = \frac{0,358 \cdot E \cdot \delta^2}{\left\{ 2 \cdot \left[ ( R_0 / a ) + 1 \right] \cdot \left[ ( b^2 / a^2 ) - 1 \right] + 1 \right\} \cdot a^2} \quad (53)$$

- for the case of the cross section with the minor axis horizontal (Figure 2):

$$\left( p_{cr} \right)_1 = \frac{0,358 \cdot E \cdot \delta^2}{\left\{ 2 \cdot \left[ ( R_0 / a ) - 1 \right] \cdot \left[ ( b^2 / a^2 ) - 1 \right] - 1 \right\} \cdot a^2} \quad (54)$$

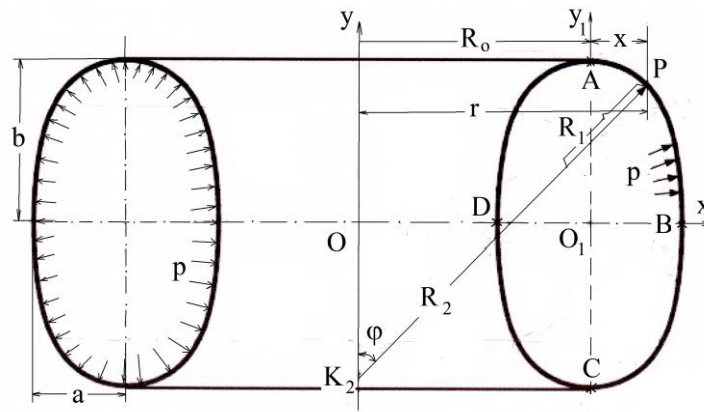


Fig. 2. Toroidal shell with elliptical cross-section with small horizontal axis ( $a < b$ ).

**Abbreviations:**

$a, b$  – semi axes of the medial ellipse of the cross section of the torus;  $p$  – working environment pressure;  $p_{cr}$  – critical pressure;  $r$  – radius of current points  $P$ ;  $u, w$  – the displacements of the meridian points along the main radii of curvature;  $x$  – current dimension;  $\delta$  – the thickness of the shell wall;  $\epsilon_1, \epsilon_2$  – specific linear deformations; angle to center of points  $P$ ;  $\nu$  – the transverse contraction coefficient of the material;  $\theta$  – rotating the meridian;  $\sigma_1, \sigma_2$  – normal meridional stress;  $\Delta r$  – variation of current radius  $r$ ;  $E$  – the modulus of longitudinal elasticity of the material;  $P$  – the position of the current points

on the medial ellipse circumference of shell;  $S$ ,  $T$  – meridional and annular unitary force developed by the inner working pressure;  $R_0$  – the radius of the position of the ellipse axis parallel to the axis of symmetry of the toroidal shell;  $R_1$ ,  $R_2$  – main radii of curvature;  $Ox$ ,  $Oy$  – the chosen reference system axe.

### 3. CONCLUSIONS

The analytical study analyzes stress and deformation states in the configuration of toroidal cross-sections of elliptical shape with two positions that can be found in industrial practice. In this regard, the two possible cases are to be considered, namely when the ellipse's large axis is in the horizontal plane, respectively in the vertical plane. The internal pressure of a working environment and its temperature effect are used as external loads. The two effects may overlap, considering the state of stress in the elastic domain of the material of the given structure.

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